9. FINITELY GENERATED ABELIAN GROUPS

§9.1. Finitely Presented Abelian Groups

The group:

$$\langle A, B, C \mid A^4, B^2, AB = BA, AC = CA, BC = CB\rangle$$

is an example of a finitely-presented abelian group, but one which is written multiplicatively. Additively we would write it as:

$$\langle A, B, C \mid 4A, 2B, A + B = B + A, A + C = C + A, B + C = C + B\rangle.$$

But if we're working entirely with abelian groups we know that the generators commute so we omit the commuting relations and use [ ] instead of $\langle\ \rangle$. We write the group simply as [A, B, C | 4A, 2B]

We denote the abelian group generated by $X_1, \dots, X_n$ subject to the relators $R_1, \dots, R_m$ by

$$[X_1, \dots, X_n \mid R_1, \dots, R_m].$$

The relators are written additively.

Now a typical relator, R_i, can be written in the form $a_{i1}X_1 + \dots + a_{in}X_n$ where the a_{ij} form an $m \times n$ matrix of integers A. Since the names of the generators are not important, and the number of them is the same as the number of columns of A, we can recover the presentation from just the integer matrix A.

For any $m \times n$ integer matrix A = (a_{ij}), [A] denotes the abelian group on n generators $X_1, \ldots, X_n$, subject to the m relations:

$$\begin{aligned} &a_{11}X_1 + \ldots + a_{1n}X_n = 0 \\ &a_{21}X_1 + \ldots + a_{2n}X_n = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &a_{m1}X_1 + \ldots + a_{mn}X_n = 0 \end{aligned}$$

Where the matrix is written in terms of its components we omit the usual matrix parentheses.

Example 1:

$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \\ 2 & 2 & 2 \end{bmatrix}$ denotes the abelian group

[A, B, C | 8A, 8B, 8C, 2A + 2B + 2C].

Essentially a finitely-presented abelian group is a system of homogeneous linear equations, but with integer coefficients. The important difference between these and those that arise in linear algebra is that here, division is not permitted. For example an element, x, of order 8 satisfies $8x = 0$ which in linear algebra would imply that $x = 0$. But that’s because in linear algebra the coefficients come from a field while for abelian groups they’re integers. We can only divide by those integers that have integer inverses under multiplication, that is, ±1.

Example 2: $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ denotes the abelian group\

$$[X, Y, Z \mid 8X, 8Y, 8Z].$$

This is clearly a direct sum of cyclic groups, each of order 8 and so the group is isomorphic to $\mathbb{Z}_8 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_8$.

Where the matrix is diagonal we can read off, from the diagonal elements, the nature of the corresponding abelian group as a direct sum of cyclic groups.

Example 3: $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ denotes the abelian group

$$[X, Y, Z \mid 8X, 8Y].$$

Strictly speaking we should have written down the third relator, 0Z, representing the relation $0Z = 0$, but this is clearly redundant. The last generator has infinite order, so the group is isomorphic to $\mathbb{Z}_8 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}$.

So where the diagonal entry is 0 the corresponding direct summand is $\mathbb{Z}$ (not $\mathbb{Z}_0$). In the above example the third row is superfluous so we can write:

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cong \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \end{bmatrix} \cong \mathbb{Z}_8 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}.$$

§9.2. Elementary Row Operations

We're in a similar situation to that part of linear algebra that deals with the solution of systems of linear equations. Remember the powerful role played by the elementary row operations in the solution of such systems and the part they play in the Gaussian algorithm.

Let's review the three types of elementary row operations.

$R_i \leftrightarrow R_j$: swap rows *i*, *j*

This is equivalent to swapping a pair of equations in our system and, just as in linear algebra, this is permissible. The new system is equivalent to the original one and so the groups are isomorphic.

Example 4:

$$\begin{bmatrix} 8 & 6 & 5 \\ 3 & 8 & -2 \\ 1 & 0 & -3 \end{bmatrix} \cong \begin{bmatrix} 3 & 8 & -2 \\ 8 & 6 & 5 \\ 1 & 0 & -3 \end{bmatrix}$$

where we've swapped R_1 and R_2.

$R_i \div k$: Divide row *i* by *k* (*k* = ±1 only)

This is where our abelian group situation differs from the linear algebra one. Our 'scalars' here are integers and division is not generally permitted. In fact the only values of *k* for which this operation is permissible are $k = \pm 1$.

R$_i$ – *k*R$_j$: subtract *k* times row *j* from row *i* (*k* any integer)

This is the most useful of all the elementary row operations in linear algebra and so it is here. Of course in our context only integer values of *k* can be used here.

Example 5:

Let G = $\begin{bmatrix} 8 & 6 & 5 \\ 3 & 8 & -2 \\ 1 & 0 & -3 \end{bmatrix}$.

Subtracting twice row 2 from row 1 we get

G ≅ $\begin{bmatrix} 2 & -10 & 9 \\ 3 & 8 & -2 \\ 1 & 0 & -3 \end{bmatrix}$.

Swapping rows 1 and 3 we get G ≅ $\begin{bmatrix} 1 & 0 & -3 \\ 3 & 8 & -2 \\ 2 & -10 & 9 \end{bmatrix}$.

Now, mimicking the Gaussian algorithm, we can subtract 3 times row 1 from row 2 and twice row 1 from row 3 to get 0's underneath the 1 in the first column.

G ≅ $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 8 & 7 \\ 0 & -10 & 15 \end{bmatrix}$.

Now, adding row 2 to row 3 (that is R$_3$ – (–)R$_2$) we get

G ≅ $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 8 & 7 \\ 0 & -2 & 22 \end{bmatrix}$. We write this as R$_3$ + R$_2$, though adding a multiple of a row is not a new operation. Note here that only the first named row gets changed. So $R_2 + R_3$ is a different operation to $R_3 + R_2$.

Now we can swap rows 2 and 3 to get $G \cong \begin{bmatrix} 1 & 0 & -3 \\ 0 & -2 & 22 \\ 0 & 8 & 7 \end{bmatrix}$.

Now, with $R_3 + 4R_2$ we get $G \cong \begin{bmatrix} 1 & 0 & -3 \\ 0 & -2 & 22 \\ 0 & 0 & 95 \end{bmatrix}$.

If we could reach a diagonal matrix we would have identified the group as a direct sum of cyclic groups. But here this seems to be about as far as we can go. Any further elementary row operations would only make the matrix more complicated – less like a diagonal matrix. We need additional operations.

§9.3. Elementary Column Operations

Elementary row operations convert a set of homogeneous linear equations into an equivalent set for the same set of variables. Once we start using column operations we begin to change the variables. But if we're only interested in the structure of the group, up to isomorphism, we can use elementary column operations to produce a simpler set of equations on a different, but equivalent, generating set.

The simplest case would be that of swapping two columns. The effect is to swap the corresponding generators. The groups described by the presentations will be isomorphic.

Example 6: $\begin{bmatrix} 3 & 3 & 6 \\ 8 & 4 & 0 \\ 0 & 12 & 12 \end{bmatrix}$

$\cong$ [A, B, C | 3A + 3B +6C = 8A + 4B = 12B + 12C = 0]
$\cong$ [A, B, C | 3A + 3C +6B = 8A + 4C = 12B + 12C = 0]
$\cong$ [A, B, C | 3A + 6B +3C = 8A + 4C = 12B + 12C = 0]

$$\cong \begin{bmatrix} 3 & 6 & 3 \\ 8 & 0 & 4 \\ 0 & 12 & 12 \end{bmatrix}$$

The effect of swapping the two generators B and C is to swap two columns of the integer matrix of the presentation.

Equally simple is an operation of the form $C_j \times -1$ which changes the sign of every entry in a given column. If X_j is the corresponding generator this corresponds to replacing X_j by $-X_j$.

When it comes to subtracting an integer multiple of one column from another the effect on the generators is a little less obvious. Consider the following example:

Example 7:

$$\begin{bmatrix} 3 & 6 & 3 \\ 8 & 17 & 4 \\ 0 & 5 & 2 \end{bmatrix}$$

= [X_1, X_2, X_3 | $3X_1 + 6X_2 + 3X_3$, $8X_1 + 17\,X_2 + 4X_3$,
$5X_2 + 2X_3 = 0$]

Define $X_1' = X_1 + 2X_2$. Clearly the group is generated by $\{X_1', X_2, X_3\}$ since $X_1 = X_1' - 2X_2$.

Expressing the relators in terms of this new set of generators we get:

$$3(X_1' - 2X_2) + 6X_2 + 3X_3,\ 8(X_1' - 2X_2) + 17X_2 + 4X_3,\ 5X_2 + 2X_3 = 0.$$

So the group has the equivalent presentation

$$[X_1', X_2, X_3 \mid 3X_1' + 3X_3, 8X_1' + X_2 + 4X_3, 5X_2 + 2X_3] \cong \begin{bmatrix} 3 & 0 & 3 \\ 8 & 1 & 4 \\ 0 & 5 & 2 \end{bmatrix}.$$

The effect of the change of variables $X_1 \to X_1' = X_1 + 2X_2$ is the elementary column operation $C_2 - 2C_1$. Note the change of sign and the swapping of the subscripts.

If the generators are $X_1, \ldots, X_n$:

$C_i \leftrightarrow C_j$ corresponds to $X_i \leftrightarrow X_j$

$C_i \times -1$ corresponds to $X_i \to -X_i$

$C_i - kC_j$ corresponds to $X_j \to X_j + kX_i$

Theorem 1: If the integer matrix B is obtained from A by a sequence of elementary row and column operations then $[B] \cong [A]$. ☺

Example 8:

$$\begin{bmatrix} 10 & 14 & 4 \\ 12 & 16 & 8 \\ 14 & 18 & 8 \end{bmatrix} \cong \begin{bmatrix} 4 & 14 & 10 \\ 8 & 16 & 12 \\ 8 & 18 & 14 \end{bmatrix} C_1 \leftrightarrow C_3$$

$$\cong \begin{bmatrix} 4 & 2 & 2 \\ 8 & -8 & -4 \\ 8 & -6 & -2 \end{bmatrix} C_2 - 3C_1$$

$$\cong \begin{bmatrix} 4 & 2 & 2 \\ 0 & -12 & -8 \\ 0 & -10 & -6 \end{bmatrix} R_2 - 2R_1,\ R_3 - 2R_1$$

$$\cong \begin{bmatrix} 2 & 4 & 2 \\ -12 & 0 & -8 \\ -10 & 0 & -6 \end{bmatrix} C_1 \leftrightarrow C_2$$

$$\cong \begin{bmatrix} 2 & 0 & 0 \\ -12 & 24 & 4 \\ -10 & 20 & 4 \end{bmatrix} C_2 - 2C_1,\ C_3 - C_1$$

$$\cong \begin{bmatrix} 2 & 0 & 0 \\ 0 & 24 & 4 \\ 0 & 20 & 4 \end{bmatrix} R_2 + 6R_1,\ R_3 + 5R_1$$

$$\cong \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 24 \\ 0 & 4 & 20 \end{bmatrix} C_2 \leftrightarrow C_3$$

$$\cong \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 4 & -4 \end{bmatrix} C_3 - 6C_2$$

$$\cong \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix} R_3 - R_2$$

$$\cong \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} C_3 \times (-1)$$

We've managed to get the matrix of an equivalent presentation in diagonal form. But in terms of the new generators this is clearly a direct sum of cyclic groups, viz. $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$.

§9.4. The Fundamental Theorem of Finitely-Generated Abelian Groups

Using the elementary row and column operations we can convert every integer matrix to diagonal form, and hence we have the following theorem.

Theorem 2: Every finitely-presented abelian group is a direct sum of cyclic groups.

Proof: Let A be the presentation matrix for a finite presentation of an abelian group.

Case (1): A is 1 × 1:

Let A = (m). We can multiply by –1, if necessary, so we may assume that $m \geq 0$.

Then $[A] \cong \mathbb{Z}$ if $m = 0$ and
$\mathbb{Z}_m$ if $m > 0$.

(Of course $\mathbb{Z}_1$ is the trivial group so may be removed if it arises.)

Case (2) A = (m , 0, ... , 0) for some m:

[A] is isomorphic to the direct sum of $\mathbb{Z}_m$ and $n - 1$ copies of $\mathbb{Z}$.

Case (3) A = $\begin{pmatrix} \boldsymbol{m} \\ \mathbf{0} \\ \mathbf{...} \\ \mathbf{0} \end{pmatrix}$ **for some *m*:** Clearly [A] ≅ $\mathbb{Z}_m$.

Case (4) A is the *m* × *n* zero matrix:
Clearly [A] is isomorphic to the direct sum of *n* copies of $\mathbb{Z}$.

Case (5) The general case:
Suppose now that A ≠ 0 and has at least 2 rows and at least 2 columns. Choose a non-zero element with smallest absolute value. Permute rows and columns to bring it to the 1-1 position and, if necessary, multiply the first column by –1 to make it positive. Now subtract suitable multiples of the first row and column from the others so that all other entries in the first row and column are in the range $0 \le x < a_{11}$.

This whole process can be continued, reducing the smallest non-zero absolute value, until the matrix takes the form $(m, 0, ... , 0)$, $\begin{pmatrix} m \\ 0 \\ ... \\ 0 \end{pmatrix}$ or $\begin{pmatrix} m & 0 \\ 0 & \mathrm{B} \end{pmatrix}$ where *m* is a non-negative integer and B is an integer matrix with one less row and column.

The theorem now follows by induction on the number of generators. ✋☺

Example 9:

$$\begin{bmatrix} 9 & 6 & 7 & 5 \\ 30 & 21 & 17 & 13 \\ 18 & 15 & 7 & 5 \end{bmatrix} \cong \begin{bmatrix} 5 & 9 & 6 & 7 \\ 13 & 30 & 21 & 17 \\ 5 & 18 & 15 & 7 \end{bmatrix} \text{ permute columns}$$

$$\cong \begin{bmatrix} 5 & 9 & 1 & 7 \\ 13 & 30 & 8 & 17 \\ 5 & 18 & 10 & 7 \end{bmatrix} \; C_3 - C_1$$

$$\cong \begin{bmatrix} 1 & 5 & 9 & 7 \\ 8 & 13 & 30 & 17 \\ 10 & 5 & 18 & 7 \end{bmatrix} \text{ permute columns}$$

$$\cong \begin{bmatrix} 1 & 5 & 9 & 7 \\ 0 & -27 & -42 & -39 \\ 0 & -45 & -72 & -63 \end{bmatrix} \; R_2 - 8R_1,\ R_3 - 10R_1$$

$$\cong \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 27 & 42 & 39 \\ 0 & 45 & 72 & 63 \end{bmatrix} \; C_5 - 5C_1,\ C_3 - 9C_1,\ C_4 - 7C_1$$

$$\cong \begin{bmatrix} 27 & 42 & 39 \\ 45 & 72 & 63 \end{bmatrix} \text{ omit } \mathbb{Z}_1$$

$$\cong \begin{bmatrix} 27 & 42 & 12 \\ 45 & 72 & 18 \end{bmatrix} \; C_3 - C_1$$

$$\cong \begin{bmatrix} 12 & 27 & 42 \\ 18 & 45 & 72 \end{bmatrix} \text{ permute columns}$$

$$\cong \begin{bmatrix} 12 & 3 & 42 \\ 18 & 9 & 72 \end{bmatrix} \; C_2 - 2C_1$$

$$\cong \begin{bmatrix} 3 & 12 & 42 \\ 9 & 18 & 72 \end{bmatrix} \text{ permute columns}$$

$$\cong \begin{bmatrix} 3 & 12 & 42 \\ 0 & -18 & -54 \end{bmatrix} \; R_2 - 3R_1$$

$$\cong \begin{bmatrix} 3 & 0 & 0 \\ 0 & 18 & 54 \end{bmatrix} C_2 - 4C_1,\ C_3 - 14C_1$$

$$\cong \mathbb{Z}_3 \oplus [18 \;\; 54]$$

$$\cong \mathbb{Z}_3 \oplus [18 \;\; 0]\ C_2 - 3C_1$$

$$\cong \mathbb{Z}_3 \oplus \mathbb{Z}_{18} \oplus \mathbb{Z}.$$

The above theorem deals with **finitely-presented** abelian groups, those where there's not only a finite set of generators, but where the relations that hold between them can be deduced from a finite set of relations. What about those that are merely finitely-generated?

By adapting the above argument slightly we can show that they too are direct sums of cyclic groups. And since direct sums of finitely many cyclic groups are finitely-presented it follows that all finitely-generated abelian groups are indeed finitely-presented!

Theorem 3: Every finitely-generated abelian group is a direct sum of cyclic groups.

Proof: Suppose we have a finitely-generated abelian group G. Consider the set of all relations that hold between the generators and let the coefficients be arranged in an integer array. This will in fact be a matrix with as many columns as there are generators, but with possibly infinitely many rows. Exactly the same algorithm can be used as before. With infinitely many rows of course there'd be practical difficulties in implementing it but since all the rows can be operated on in parallel there'd be no theoretical problem. The finiteness of the number of columns means that the algorithm will terminate eventually. ✋☺

§9.5. Euler's Theorem

A ready source of finite abelian groups can be found as integers modulo *m* under multiplication. Recall that if *m* is any positive integer $\mathbb{Z}_m^{\#}$ denotes the group of all numbers from 1 to *m* that are coprime with *m*, under the operation of multiplication modulo *m*. (The coprimeness ensures the existence of inverses.)

Example 10: $\mathbb{Z}_7^{\#} = \{1, 2, 3, 4, 5, 6\} \cong \mathbb{Z}_6$;

$$\mathbb{Z}_8^{\#} = \{1, 3, 5, 7\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2;$$

$$\mathbb{Z}_{10}^{\#} = \{1, 3, 7, 9\} \cong \mathbb{Z}_4.$$

The order of $\mathbb{Z}_m^{\#}$ is denoted by $\varphi(m)$.This function φ is called the **Euler $\boldsymbol{\varphi}$ function**. (**NOTE** you pronounce 'Euler' as 'Oiler'.) It is an important function in number theory, with $\varphi(m)$ being the number of numbers from 1 to *m* that are coprime with *m*.

Lemma: (CHINESE REMAINDER THEOREM)
If *m*, *n* are coprime then for all $a, b \in \mathbb{Z}$ there exists $x \in \mathbb{Z}$ such that:

$$x \equiv a \pmod{m} \text{ and}$$
$$x \equiv b \pmod{n}.$$

Proof: Since *m*, *n* are coprime there exist integers *h*, *k* such that $1 = mh + nk$.

Let $x = a + m(b - a)h$. Clearly $x \equiv a \pmod{m}$.

Now $x = a(1 - mh) + mhb$

$$\begin{aligned} &= a(nk) + mhb \\ &= nka + (1 - nk)b \\ &= b + nk(a - b) \\ &\equiv b \pmod{n}. \end{aligned}$$ ✋☺

Theorem 4: If m, n are coprime then $\mathbb{Z}_{mn}^{\#} \cong \mathbb{Z}_m^{\#} \times \mathbb{Z}_n^{\#}$. (We use "×" here instead of '⊕' simply because we're using multiplicative notation.)

Proof: Suppose that m, n are coprime. Then $x \to (x, x)$ is a homomorphism from $\mathbb{Z}_{mn}^{\#}$ to $\mathbb{Z}_m^{\#} \times \mathbb{Z}_n^{\#}$ since x is coprime to mn if and only if it's coprime to both m and n.

The kernel of this map is trivial since, if $x \to (1, 1)$, then $x - 1$ is a multiple of both m and n and so is a multiple of mn (because m and n are coprime). The fact that this map is onto is a consequence of the Chinese Remainder Theorem (the lemma above). ✋☺

Corollary: If m, n are coprime $\varphi(mn) = \varphi(m)\,\varphi(n)$.

Theorem 5: If p is prime, $\varphi(p^n) = p^{n-1}(p - 1)$

Proof: Of the p^n numbers from 0 to $p^n - 1$ there are precisely p^{n-1} multiples of p. The remaining $p^n - p^{n-1} = p^{n-1}(p - 1)$ numbers will be precisely the ones with no factor in common with p^n. Hence $\varphi(p^n) = p^{n-1}(p - 1)$. ✋☺

Example 11: $\varphi(200) = \varphi(2^3.5^2) = 2^2(2-1)\,5^1(5-1)$

$= 4.1.5.4 = 80.$

Theorem 6: (EULER) If a is coprime with m then

$$a^{\varphi(m)} \equiv 1 \pmod{m}.$$

Proof: Suppose a is coprime with m. Then $a \in \mathbb{Z}_m{}^{\#}$. Suppose it has order n. By Lagrange's theorem n divides $\varphi(m)$. Thus $\varphi(m) = nq$ for some $q \in \mathbb{Z}$.

Now $a^{\varphi(m)} = (a^n)^q = 1^q = 1$. ✋☺

Corollary: (FERMAT) If p is prime then

$$\boldsymbol{a^p \equiv a \pmod{p}}.$$

Proof: If p divides a then LHS = RHS = 0.

Otherwise, by Euler’s theorem $a^{p-1} \equiv 1(\text{mod } p)$.

Euler's theorem can be used to calculate the remainders of certain very large numbers.

Example 12: What is the remainder on dividing 5^{1000} by 42?

Solution: $\varphi(42) = \varphi(2.3.7) = 12$ so $5^{12} = 1 \pmod{42}$.

We note that 5 is coprime to 42.

Dividing 1000 by 12 we get a remainder of 4.

$[1000 = 12 \times 83 + 4]$

So $5^{1000} = (5^{12})^{83}.5^4 = 1^{83}.5^4 = 625 = 37$.

Hence 5^{1000} leaves a remainder of 37 when divided by 42.

NOTE: To work this out directly, by calculating 5^{1000} first, would need far more computing power than is normally available.

In the decomposition of a finitely-generated group as a direct sum of cyclic groups the only finite summands we need are those whose orders are prime powers. This is because of the following theorem, which parallels Theorem 5.

Theorem 7: If m, n are coprime then $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \oplus \mathbb{Z}_n$.
Proof: Let $x = (1, 1) \in \mathbb{Z}_m \oplus \mathbb{Z}_n$.
Then $kx = (k, k) = (0, 0)$ if and only if k is both a multiple of m and n. Since m, n are coprime this requires k to be a multiple of mn and so x has order mn, the same as the order of the group $\mathbb{Z}_m \oplus \mathbb{Z}_n$ itself. Hence $\mathbb{Z}_m \oplus \mathbb{Z}_n$ is cyclic and so is isomorphic to $\mathbb{Z}_{mn}$. ✋☺

Example 13: $\mathbb{Z}_{24} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_8$. Note that we can’t split $\mathbb{Z}_8$ into $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ because $\mathbb{Z}_8$ has only one element of order 2 while $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ has 7 such elements.

§9.6. Some Important Subgroups of an Abelian Group

For an integer n and an abelian group G we define

$$n\mathbf{G} = \{ng \mid g \in \mathbf{G}\}.$$

Clearly this is a subgroup of G since $n(x + y) = nx + ny$ etc. Using multiplicative notation we would write this as

$G^n = \{g^n \mid g \in G\}$. Since this may not be a subgroup of G if G is non-abelian we don't define this unless the group is abelian.

Examples 14:

(1) If $G = \mathbb{Z}_4 \oplus \mathbb{Z}_8$, then

$2G = \{(0, 0), (0, 2), (0, 4), (0, 6), (2, 0), (2, 2), (2, 4), (2, 6)\}$
$\quad \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$ and $3G = G$.

(2) If $G = \mathbb{Z}_4 \oplus \mathbb{Z}_6$ then

$2G = \{(0, 0), (0, 2), (0, 4), (2, 0), (2, 2), (2, 4)\}$
$\quad \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$ and
$3G = \{(0, 0), (0, 3), (3, 0), (3, 3), (2, 0), (2, 3), (1, 0), (1, 3)\} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$.

Theorem 8: If G, H are abelian groups

$$n(G \oplus H) \cong nG \oplus nH.$$

Proof: The map $n(x, y) = (nx, ny)$ is an isomorphism. ✋☺

Theorem 9: $m\,\mathbb{Z}_n \cong \mathbb{Z}_d$ where $d = \dfrac{n}{\text{GCD}(m, n)}$.

Proof: $m\mathbb{Z}_n$ is clearly cyclic, generated by m, and $km = 0$ in $\mathbb{Z}_n$ if and only if $\dfrac{n}{\text{GCD}(m, n)}$ divides k. Thus m has order $\dfrac{n}{\text{GCD}(m, n)}$ and generates a cyclic group that is isomorphic to $\mathbb{Z}_{n/\text{GCD}(m, n)}$. ✋☺

Example 15:
If $G = \mathbb{Z}_{30} \oplus \mathbb{Z}_{100}$, $2G \cong \mathbb{Z}_{15} \oplus \mathbb{Z}_{50}$, $3G \cong \mathbb{Z}_{10} \oplus \mathbb{Z}_{100}$, $6G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_{50}$ and $28G \cong \mathbb{Z}_{15} \oplus \mathbb{Z}_{25}$.

Another useful subgroup, for each positive integer n, is $\mathbf{G[}\boldsymbol{n}\mathbf{] = \{}\boldsymbol{g} \in \mathbf{G} \mid \boldsymbol{ng} = \mathbf{0\}}$. It consists of those elements of G whose order divides n and it's clearly a subgroup of G since $nx = 0$ and $ny = 0$ imply $n(x + y) = 0$, etc.

We use the same notation if the abelian group is written multiplicatively, but here we would define it as $\mathbf{G[}\boldsymbol{n}\mathbf{] = \{}\boldsymbol{g} \in \mathbf{G} \mid \boldsymbol{g}^{\boldsymbol{n}} = \mathbf{1\}}$. Again, for a non-abelian group it isn't usually a subgroup and so we don't define it unless the group is abelian.

Examples 16:
(1) If $G = \mathbb{Z}_4 \oplus \mathbb{Z}_8$,
$G[2] = \{(0, 0), (0, 4), (2, 0), (2, 4)\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and
$G[3] = 0$.
(2) If $G = \mathbb{Z}_4 \oplus \mathbb{Z}_6$, $G[2] = \{(0, 0), (0, 3), (2, 0), (2, 3)\}$
$\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and
$G[3] = \{(0, 0), (0, 2), (0, 4)\} \cong \mathbb{Z}_3$.
(3) If $G = \mathbb{Z}_{20}^{\#}$ then $G^2 = \{1, 3^2, 7^2, 9^2, 11^2, 13^2, 17^2, 19^2\}$
$= \{1, 3^2, 7^2, 9^2\}$ since $(-x)^2 = x^2$
$= \{1, 9\} \cong \mathbb{Z}_2$ and
$G[2] = \{1, 9, 11, 19\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Theorem 10: If G, H are abelian groups

$$(\mathrm{G} \oplus \mathrm{H})[n] = \mathrm{G}[n] \oplus \mathrm{H}[n].$$

Proof: This is because $k(x, y) = 0$ if and only if $kx = 0$ in G and $ky = 0$ in H. ✋☺

Theorem 11: $\mathbb{Z}_m[n] \cong \mathbb{Z}_{\mathrm{GCD}(m,\, n)}$.

Proof: Suppose $k \in \mathbb{Z}_m[n]$. Then $nk = 0$ in $\mathbb{Z}_m$ and so m divides nk. Hence $\dfrac{m}{\mathrm{GCD}(m, n)}$ divides k. Thus $\mathbb{Z}_m[n]$ is a cyclic group generated by $\dfrac{m}{\mathrm{GCD}(m, n)}$ and so is isomorphic to $\mathbb{Z}_{\mathrm{GCD}(m,\, n)}$. ✋☺

Example 17:

If $\mathrm{G} = \mathbb{Z}_{30} \oplus \mathbb{Z}_{100}$, $\mathrm{G}[2] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathrm{G}[3] \cong \mathbb{Z}_3$,
$\mathrm{G}[6] \cong \mathbb{Z}_6 \oplus \mathbb{Z}_2$ and $\mathrm{G}[28] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$.

§9.7. The Order Profile of a Finite Abelian Group.

Once a finite group G has been written as a direct sum of cyclic groups the numbers of elements of each order can be easily determined. This is because we can easily identify the subgroups G[n] for each n and hence recover the order information. A table that lists the numbers of elements of each order is called the **order profile** of the group.

Since the order of G[*n*] is the number of elements whose order divides *n*, we can count the number of elements of order *n* as follows:

# elements of order n in G

$$= |G[n]| - \sum_{d|n,\, d<n} \#\textit{elements of order } d$$

Example 18: Find the order profile of G = $\mathbb{Z}_4 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_9$.
Solution: Since 36G = 0 the order of each element divides 36.
We list the subgroups G[*n*] and their orders:

n	**1**	**2**	**3**	**4**	**6**
G[*n*]	1	$\mathbb{Z}_2\oplus \mathbb{Z}_2$	$\mathbb{Z}_3\oplus \mathbb{Z}_3$	$\mathbb{Z}_4\oplus \mathbb{Z}_2$	$\mathbb{Z}_2\oplus \mathbb{Z}_6\oplus \mathbb{Z}_3$
|G[*n*]|	1	4	9	8	36

n	**9**	**12**	**18**	**36**
G[*n*]	$\mathbb{Z}_3\oplus \mathbb{Z}_9$	$\mathbb{Z}_4\oplus \mathbb{Z}_6\oplus \mathbb{Z}_3$	$\mathbb{Z}_2\oplus \mathbb{Z}_6\oplus \mathbb{Z}_9$	G
|G[*n*]|	27	72	108	216

So the order profile is:

order	**number**	
1	1	
2	3	= 4 – 1
3	8	= 9 – 1
4	4	= 8 – 3 – 1
6	24	= 36 – 8 – 3 – 1

9	18	= 27 – 8 – 1
12	32	= 72 – 24 – 4 – 8 – 3 – 1
18	54	= 108 – 18 – 24 – 8 – 3 – 1
36	72	= 216 – 54 – 32 – 18 – 24
TOTAL	**216**	– 4 – 8 – 3 – 1

The above process can be reversed. For a finite abelian group, knowing the number of elements of each order is sufficient to identify the group, up to isomorphism. (This can’t be done with non-abelian groups as there are non-isomorphic groups with the same order profile.)

If p is a prime and G is a finite group then the **Sylow *p*-subgroup of G** is the set of all elements whose order is a power of p. It is named after the Norwegian mathematician Ludwig Sylow [1832 – 1918]. We denote it by **Syl$_p$(G)**. (For non-abelian groups this set is not usually a subgroup and Sylow subgroups are defined differently.)

Example 19: The Sylow 2-subgroup of $\mathbb{Z}_{100}$ is $\{0, 25, 50, 75\} = \langle 25\rangle$ which is isomorphic to $\mathbb{Z}_4$.

Theorem 12: Every finite abelian group is the direct sum of its Sylow subgroups.

Proof: We can write every finite abelian group as a direct sum of cyclic groups. Every cyclic group can be broken up as a direct sum of cyclic p-groups, for various primes *p*. Collecting all those for a particular prime *p* we get the corresponding Sylow *p*-subgroup. Hence the group can be written as a direct sum of its Sylow subgroups.

Example 20: $\mathbb{Z}_{25}{}^{\#}$ has order 20 and an element of order 4, so it is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_5$.
Hence $\mathbb{Z}_{100}{}^{\#} \oplus \mathbb{Z}_{50}{}^{\#} \cong \mathbb{Z}_4{}^{\#} \oplus \mathbb{Z}_{25}{}^{\#} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5$.
So $\mathrm{Syl}_2(\mathbb{Z}_{100}{}^{\#}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$ and $\mathrm{Syl}_5(\mathbb{Z}_{100}{}^{\#}) \cong \mathbb{Z}_5$.

Theorem 13: If G, H are finite abelian groups then

$$\mathrm{Syl}_p(G \oplus H) = \mathrm{Syl}_p(G) \oplus \mathrm{Syl}_p(H).$$

Example 21:
$\mathrm{Syl}_2(\mathbb{Z}_{100} \oplus \mathbb{Z}_{50}) = \mathrm{Syl}_2(\mathbb{Z}_{100}) \oplus \mathrm{Syl}_2(\mathbb{Z}_{50}) \cong \langle 25\rangle \oplus \langle 25\rangle$.
We have to be a little careful here because these two direct summands are not equal. They are both generated additively by 25 but in different groups. The first is isomorphic to $\mathbb{Z}_4$, while the second is isomorphic to $\mathbb{Z}_2$. Hence $\mathrm{Syl}_2(\mathbb{Z}_{100} \oplus \mathbb{Z}_{50}) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$.

Example 22: Find the Sylow 2-subgroup of $\mathbb{Z}_{100}{}^{\#}$.
Solution: Since $\mathbb{Z}_{100}{}^{\#} \cong \mathbb{Z}_4{}^{\#} \times \mathbb{Z}_{25}{}^{\#}$

$$\mathrm{Syl}_2(\mathbb{Z}_{100}{}^{\#}) \cong \mathrm{Syl}_2(\mathbb{Z}_4{}^{\#}) \times \mathrm{Syl}_2(\mathbb{Z}_{25}{}^{\#}).$$

Now $|\mathbb{Z}_4{}^{\#}| = \varphi(4) = 2$ while $|\mathbb{Z}_{25}{}^{\#}| = \varphi(25) = 20$.
Hence $|\mathrm{Syl}_2(\mathbb{Z}_{25}{}^{\#})| = 4$ and, of course $|\mathrm{Syl}_2(\mathbb{Z}_4{}^{\#})| = 2$.
It follows that $|\mathrm{Syl}_2(\mathbb{Z}_{100}{}^{\#})| = 8$.
So $\mathrm{Syl}_2(\mathbb{Z}_{100}{}^{\#}) \cong \mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
We can decide which one by finding the number of elements of order 2.

If $x \in \mathbb{Z}_{100}^{\#}$ has order 2 then $(x-1)(x+1) \equiv 0 \pmod{100}$. Clearly x must be odd and so $(x-1)(x+1) \equiv 0 \pmod{25}$. Now 25 must divide the product, and 5 can't divide each of the factors because they are only 2 apart. So 25 must divide one of the factors.

The only possibilities are 1, 49, 51 and 99 and so there are 3 elements of order 2 which means that $\mathrm{Syl}_2(\mathbb{Z}_{100}^{\#}) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$. We just need to find an element of order 4. Clearly 7 will be a candidate and 7 times each of the elements of order 2 will give the other 3 elements of order 4. So $\mathrm{Syl}_2(\mathbb{Z}_{100}^{\#}) = \{1, 7, 43, 49, 51, 57, 93, 99\}$.

Given the order profile of a finite abelian group we can identify it as a direct sum of cyclic groups.

Example 23: A certain abelian group has order 216 = 8 × 27. It could be any one of the following nine possibilities:

$\mathbb{Z}_8 \oplus \mathbb{Z}_{27}$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{27}$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{27}$	$\mathbb{Z}_8 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3$
$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3$
$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$	

Suppose we're given its order profile:

order	**number**
1	1
2	3
3	8
4	4
6	24
9	18
12	32
18	54
36	72
TOTAL	**216**

Since G has elements of order 4, but none of order 8,

$$\mathrm{Syl}_2(G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4.$$

Similarly G[3] has order 9 and so $\mathrm{Syl}_3(G)$ must be the direct sum of two cyclic subgroups and so has to be $\mathbb{Z}_3 \oplus \mathbb{Z}_9$. Thus $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9$.

Note that this is not in the form that we began with in the previous example, but since $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_6$ it can be easily brought to that form if we desire.

If a Sylow *p*-subgroup has order p^4 and G[*p*] has order p^2 we know that it has two cyclic direct summands in its decomposition, but we don't know whether it is $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$. In such a case we'd need to examine elements of higher order. In the first case G[p^2] would have order p^4 while in the second case it would have order p^3.

Example 24: Which abelian group has the following order profile?

order	number
1	1
2	7
4	24
8	32
TOTAL	**64**

So G[2] has order $8 = 2^3$ so is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Thus there are 3 cyclic summands in the direct sum decomposition.

Since G[4] has order $1 + 7 + 24 = 32 = 2^5$ it's isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$, so one of the cyclic summands is just $\mathbb{Z}_2$. Since G[8] has order $64 = 2^6$ it must be $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8$. But clearly G[8] = G so

$$G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8.$$

§9.8. The Alexander Group of a Knot

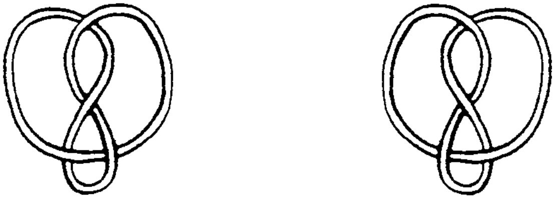
There are many places throughout mathematics where finitely-generated abelian groups arise in a very natural way. One of these is that part of topology that studies knots.

What motivates knot theorists is not the desire to come up with a better knot for tying things (even though the knots we tie in ropes, such as the granny knot, are indeed knots in the knot theorist's sense).

Last century chemists believed that space was knotted and that this was somehow connected to the chemical properties of a substance. This caused a flurry of activity in the area. Later it proved not to be the case and so for many decades knot theory was considered as a bit of a curiosity. But in the last twenty years there's been a resurgence of activity. This is partly because new methods were developed (and in the first instance by a physicist) and partly because physicists and biologists have begun to see knotted-ness in the things they study such as molecular flows and DNA.

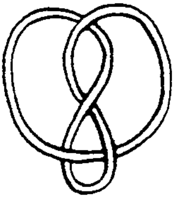
A **knot** is a closed curve in $\mathbb{R}^3$ that doesn't intersect itself. The knots we tie have two loose ends. But in order to keep the integrity of a knot, so that it doesn't change into another, we need to keep the ends far apart, or better still, we simply join them together.

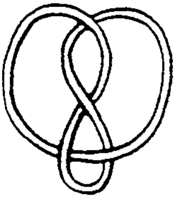
Example 25: The figure 8 knot and its mirror image are:



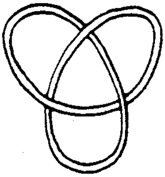
Two knots are **equivalent** if one can be deformed into the other without breaking it open.

Example 26: These two figure 8 knots are equivalent. The proof is in the doing. Take a piece of string, tie the knot and then join the ends together. Manipulate the knot, without untying, so it looks like the other.

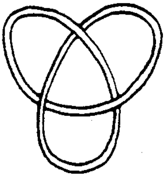




But the figure 8 knot is not equivalent to the trefoil knot that's shown at the right. This is not simply because of a different number of crossings. For example in the right-hand picture of the figure 8 knot above, if we change the over/under nature of the middle crossing it becomes equivalent to this trefoil even though it would still have four crossings. (You can demonstrate this with an actual piece of string!)



In fact there are two, distinct, trefoil knots, the other being



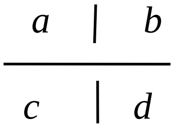
Unlike the figure 8 knot, no amount of manipulation can change one trefoil knot into the other.

It is easy to prove that two knots are equivalent. You just have to change one into the other with a oiece of string (or provide a series of drawings to illustrate the process). But how do you prove that two knots are inequivalent? The answer is to construct **invariants**, that is, mathematical objects that you can prove will remain the same as a knot is manipulated. If two knots have different values of such an invariant they cannot be equivalent.

The **Alexander group A(K)** of a knot is an abelian group that is just such an invariant. If two knots have non-isomorphic groups they're inequivalent (though if they have isomorphic Alexander groups they may still be inequivalent).

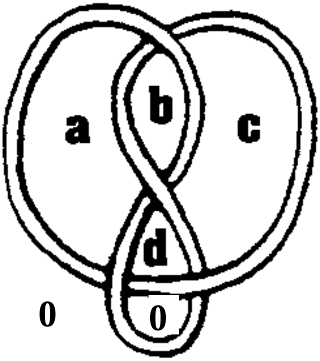
Suppose a knot has a projection with n crossings. Regarding this as a map on the sphere (the outside being counted as a region) there are n vertices and n edges. By Euler's theorem: V + F – E = 2 where V = E = n. There are thus n + 2 'faces' or regions. We assign the value 0 to two adjacent faces (usually we choose one of these to be the outside region of the knot). We then assign a generator to each of the remaining faces. These are the generators of A(K). There are n relations, one for each crossing.

If the regions surrounding a crossing are a, b, c, d, with a, b one side of the overpass and c, d on the other



the corresponding relation is $a + b = c + d$.

Example 27: For the figure-eight knot the Alexander Group is $A(K) = [a, b, c, d \mid a + b = c, a + d = b + c, a = d, c + d = 0]$



$$\cong \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cong \mathbb{Z}_5.$$

You can find further information on Alexander Groups, as well as other invariants, in my notes on *Topology*.

EXERCISES FOR CHAPTER 9

EXERCISE 1: For each of the following statements determine whether it is true or false.

(1) All cyclic groups are abelian.

(2) All abelian groups are cyclic.

(3) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is a cyclic group.

(4) $\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \cong \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix}$.

(5) $\begin{bmatrix} 2 & 2 & 0 \\ 0 & 8 & 0 \end{bmatrix} \cong \begin{bmatrix} 2 & 2 \\ 0 & 8 \end{bmatrix}$.

(6) $\mathbb{Z}_8 \oplus \mathbb{Z}_{10} \cong \mathbb{Z}_{80}$.

(7) $\mathbb{Z}_8 \oplus \mathbb{Z}_{11} \cong \mathbb{Z}_{88}$.

(8) Every finitely generated abelian group is a direct sum of cyclic groups of prime power order.

(9) $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ has more elements of order 2 than $\mathbb{Z}_2 \oplus \mathbb{Z}_4$.

(10) Every non-trivial subgroup of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$.

EXERCISE 2: Write down the relation matrix for the abelian group:

$$[A, B, C \mid 8A = 2B = 8C = 4A = 10B + 12C = 0]$$

EXERCISE 3: Write down the relation matrix for the abelian group $\mathbb{Z}_{16} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$.

EXERCISE 4: Write $\mathbb{Z}_{3000}$ as a direct sum of cyclic groups of prime power order.

EXERCISE 5: Write the abelian group

$$[A, B \mid 4A + 4B = 6A + 8B]$$

as a direct sum of cyclic groups.

EXERCISE 6: Write the abelian group

$$[A, B, C \mid 2A + 2B + 2C = 0]$$

as a direct sum of cyclic groups.

EXERCISE 7: Write the following abelian group as a direct sum of cyclic groups of prime power order:

$$\begin{bmatrix} 11 & 22 & 13 \\ 14 & 25 & 16 \\ 19 & 50 & 23 \end{bmatrix}.$$

SOLUTIONS FOR CHAPTER 9

EXERCISE 1:
(1) TRUE; (2) FALSE; (3) TRUE; (4) TRUE; (5) FALSE; (6) FALSE; (7) TRUE; (8) FALSE (infinite ones are not); (9) TRUE; (10) TRUE.

EXERCISE 2: $\begin{bmatrix} 8 & -2 & 0 \\ 4 & 0 & -8 \\ 0 & 10 & 12 \end{bmatrix}$.

EXERCISE 3: $\begin{bmatrix} 16 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

EXERCISE 4: $\mathbb{Z}_{125} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_3$

EXERCISE 5: The relation matrix is $\begin{bmatrix} 4 & 4 \\ 6 & 8 \end{bmatrix} \cong \begin{bmatrix} 4 & 4 \\ 2 & 4 \end{bmatrix}$ $\cong \begin{bmatrix} 2 & 4 \\ 0 & -4 \end{bmatrix} \cong \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix} \cong \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$.

EXERCISE 6: The group is [2, 2, 2] ≅ [2, 0, 0] $\cong \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$.

EXERCISE 7:

$$\begin{bmatrix} 11 & 22 & 13 \\ 14 & 25 & 16 \\ 19 & 50 & 23 \end{bmatrix} \cong \begin{bmatrix} 11 & 22 & 13 \\ 3 & 3 & 3 \\ 8 & 28 & 10 \end{bmatrix}$$

$$\cong \begin{bmatrix} 3 & 3 & 3 \\ 11 & 22 & 13 \\ 8 & 28 & 10 \end{bmatrix}$$

$$\cong \begin{bmatrix} 3 & 3 & 3 \\ 2 & 13 & 4 \\ 2 & 22 & 4 \end{bmatrix}$$

$$\cong \begin{bmatrix} 2 & 13 & 4 \\ 3 & 3 & 3 \\ 2 & 22 & 4 \end{bmatrix}$$

$$\cong \begin{bmatrix} 2 & 13 & 4 \\ 1 & -10 & -1 \\ 0 & 9 & 0 \end{bmatrix}$$

$$\cong \begin{bmatrix} 1 & -10 & -1 \\ 2 & 13 & 4 \\ 0 & 9 & 0 \end{bmatrix}$$

$$\cong \begin{bmatrix} 1 & -10 & -1 \\ 0 & 33 & 6 \\ 0 & 9 & 0 \end{bmatrix}$$

$$\cong \begin{bmatrix} 1 & 0 & 0 \\ 0 & 33 & 6 \\ 0 & 9 & 0 \end{bmatrix}$$

$$\cong \begin{bmatrix} 33 & 6 \\ 9 & 0 \end{bmatrix}$$

$$\cong \begin{bmatrix} 9 & 0 \\ 33 & 6 \end{bmatrix}$$

$$\cong \begin{bmatrix} 9 & 0 \\ 6 & 6 \end{bmatrix}$$

$$\cong \begin{bmatrix} 6 & 6 \\ 9 & 0 \end{bmatrix}$$

$$\cong \begin{bmatrix} 6 & 6 \\ 3 & -6 \end{bmatrix}$$

$$\cong \begin{bmatrix} 3 & -6 \\ 6 & 6 \end{bmatrix}$$

$$\cong \begin{bmatrix} 3 & -6 \\ 0 & 18 \end{bmatrix}$$

$$\cong \begin{bmatrix} 3 & 0 \\ 0 & 18 \end{bmatrix}$$

$$\cong \mathbb{Z}_3 \oplus \mathbb{Z}_{18}$$

$$\cong \mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_2.$$